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Note

A Numerical Method for Creep Deformation of Solids

This note is concerned with extension of the procedure described by Andrews and Hancock [1] to time-dependent problems in which motion is slow enough that inertial forces are negligible. Time dependence of the solution may arise from stress relaxation of the material and from time-dependent boundary conditions. The procedure involves advancing through time in finite steps and iterating to achieve stress equilibrium at each step in time. It is discussed in terms of a Maxwellian viscoelastic model that can be generalized to nonlinear cases.

Stress components σ_{ij} are decomposed into stress deviators and pressure:

$$\sigma_{ij} = s_{ij} - P\delta_{ij} .$$

Pressure is uniquely determined by volume, but stress deviator components obey a stress relaxation law, which in differential form is

$$ds_{ij} = 2\mu\, de_{ij} - s_{ij}\, dt/\tau$$

where e_{ij} is strain deviator, μ is shear modulus, and τ is relaxation time. If τ is constant this is linear Maxwellian viscoelasticity. In nonlinear cases τ is a function of stress. To have a properly covariant description, τ should be expressed as a function of stress invariants.

The finite difference equation used to advance stress in one zone from time step n to step $n + 1$ is derived as follows. Let $(\dot{e}_{ij})^{n+1/2}$ be the strain deviator rate, found from velocities, for that zone for advancing from time step n to a particular iteration at time step $n + 1$. A finite difference analog of the differential equation above is

$$\begin{aligned}(s_{ij})^{n+1} = {} & (s_{ij})^n + 2\mu(\dot{e}_{ij})^{n+1/2}\,\Delta t \\ & - 1/2[(s_{ij})^{n+1} + (s_{ij})^n]\,\frac{\Delta t}{\tau}.\end{aligned}$$

This may be rearranged to get an explicit equation

$$\begin{aligned}(s_{ij})^{n+1} = {} & \Big[(s_{ij})^n + 2\mu(\dot{e}_{ij})^{n+1/2}\,\Delta t \\ & - 1/2(s_{ij})^n\,\frac{\Delta t}{\tau}\Big]\Big/\Big(1 + 1/2\,\frac{\Delta t}{\tau}\Big).\end{aligned}$$

This equation is stable for all values of Δt and is accurate to second order in $\Delta t/\tau$.

An iteration must be performed to converge to stress equilibrium at time step $n + 1$. Solution of the above equation for all zones constitutes step 4 of the iteration outlined in [1].

To proceed through the next iteration at time step $n + 1$, stress components just calculated are used to find the unbalanced force on each grid point (step 1 of the iteration). Then, grid points are displaced in the direction of this force to go from positions in the previous iteration at time step $n + 1$ to positions in the current iteration at time step $n + 1$ (step 2). The velocities of grid points from time step n to the current iteration in time step $n + 1$, are found. From these velocities strain rates are found (step 3), and then the stress calculation may be repeated.

In the case of nonlinear stress relaxation, the relaxation time τ should be evaluated from invariants of the stress tensor averaged at the new and old times. In this average one may use stress at time step n and stress from the previous iteration at time step $n + 1$.

This procedure has been used in a problem with a cubic creep law. The iteration behaved in a reasonable way.

To check the accuracy of the method a problem was done with a linear viscoelastic material in an infinite half space, with a pressure applied to the surface. The x-axis extends into the medium and the surface is at $x = 0$. The material has been at rest with no pressure on the surface at all times up to $t = 0$. At $t = 0$ the pressure

$$p = P \cos \alpha y$$

is suddenly applied to the surface and is held constant thereafter. The analytic solution is found by applying Bland's correspondence principle [2] to the elastic solution [3]. The components of displacement are

$$u = (P/2\mu\alpha)\, e^{-\alpha x} \cos \alpha y[-(\mu/k)(1 - a)\, e^{-at/\tau} + (\mu/k) + 1 + \alpha x + (1 + \alpha x)(t/\tau)]$$

and

$$v = -(P/2\mu\alpha)\, e^{-\alpha x} \sin \alpha y[-(\mu/k)(1 - a)\, e^{-at/\tau} + (\mu/k) - \alpha x - \alpha x(t/\tau)],$$

where k is bulk modulus and

$$a^{-1} = 1 + \mu/(3k).$$

At $t = 0$ these expressions are the solution for the elastic case [3]. Note that in the elastic case the horizontal displacement reverses direction at a depth

$$\alpha x = 1 - 2\nu,$$

while in the viscoelastic solution the horizontal velocity at late times is in the same direction at all depths.

In the numerical test case we will choose $\tau = 1$, $\alpha = 1$, $2\mu = 1$, $\nu = 0.2$. The numerical method is valid for large displacements, but the analytic solution holds only for small displacements. To keep displacement small we choose $P = 10^{-4}$. Displacements are multiplied by 10^4 in the figures.

In the finite difference calculation eight zones are used in a half wavelength of the pressure variation, and the region considered is 12 zones deep. Zones are approximately square. The pressure was suddenly applied at time zero, and the calculation proceeded for one relaxation time. The time step used was $\tau/10$. In each time step 200 iterations were performed to approach stress equilibrium. The number of iterations required for long wavelength components to converge increases when finer zoning is used. It is proportional to the square of the number of zones in one dimension.

Displacements calculated after the first time step are shown in Fig. 1. This is approximately the elastic solution expected for instantaneous displacement. Variation of each component of displacement in the direction parallel to the surface is sinusoidal, as it should be, within one percent. Time dependence of the

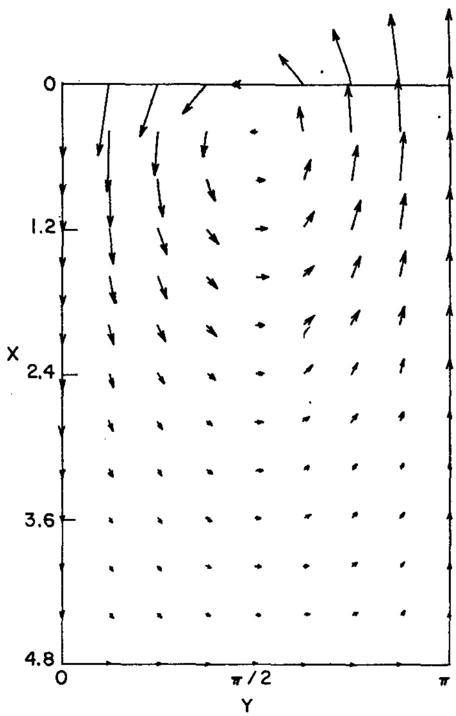

FIG. 1. Displacement field in the demonstration problem after the first step in time.

x-component of displacement at $y = 0$ is shown for four different depths in Fig. 2. Symbols show calculated values and the solid curves are the analytic solution. In

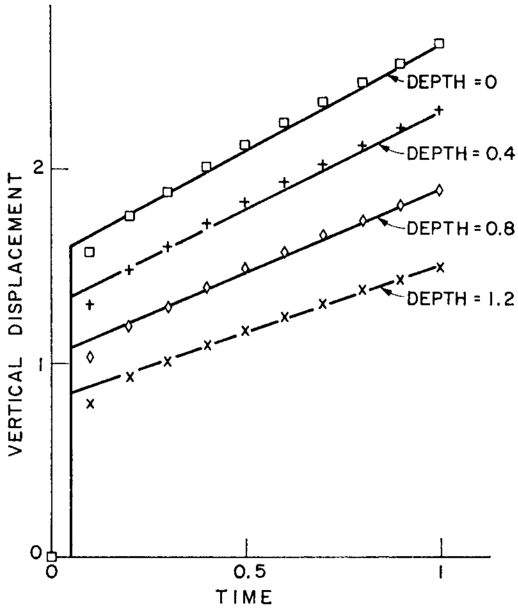

FIG. 2. Vertical displacement at four different depths as a function of time. Symbols are calculated values. Solid curves are the analytic solution.

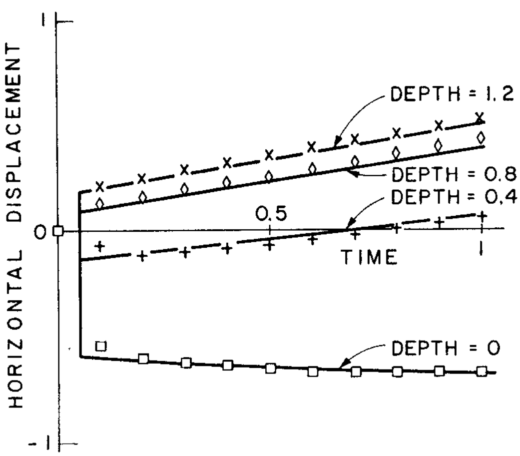

FIG. 3. Horizontal displacement at four different depths or a function of time. Symbols are calculated values. Solid curves are the analytic solution.

the first time step errors are six percent of the maximum displacement. This error could have been reduced by using a larger number of iterations. The deviation from stress equilibrium is partly corrected in the next time step, where errors are less than two percent of maximum displacement. Time dependence of the y-component of displacement at $y = \pi/2$ is shown in Fig. 3.

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